Nonresonant Diffusion of Particles in Stochastic Fields

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A Fokker-Planck equation for the non-resonant scattering of particles by general weakly turbulent electromagnetic fields is derived using a simple quantum-mechanical method. The equation is compared with the corresponding weak turbulence equation as given e. g. in Kadomtsev's book and two applications are discussed.

1. Introduction

To describe the diffusion of charged particles in turbulent electromagnetic fields it is often necessary to go beyond the "quasi-linear" approximation (second order in the field amplitude). This is the case if a) the particles under consideration do not resonate with the modes of the field, $\omega - \boldsymbol{k} \cdot \boldsymbol{v} \neq 0$ (or $\omega - k_{\|} v_{\|} - n \Omega \neq 0$ in a static magnetic field), and b) time and space variations of the average field quantities are small, i. e. the system is sufficiently stationary and homogeneous, such that the second order adiabatic terms [see e. g. Ref. [1], Eq. (I. 59)] are unimportant. Two examples, discussed in section 4, are ions moving in a field of ion sound waves and electrons in a radiation field.

The classical procedure to obtain a Fokker-Planck equation is to derive from the equation of motion of the particles the diffusion and friction coefficients $\Delta \boldsymbol{v}^2$, $\Delta \boldsymbol{v}$ defined by

$$\Delta \boldsymbol{v}^2 \equiv \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \langle (\boldsymbol{v}(t + \Delta t) - \boldsymbol{v}(t))^2 \rangle$$

and similar for $\Delta \boldsymbol{v}$.

This has been done for the resonant [2] diffusion of charged particles by STURROCK [3], for the nonresonant diffusion (fourth order in the field amplitude) by MACMAHON and DRUMMOND [4]. Both papers consider only electrostatic oscillations and the velocity space diffusion parallel to a strong magnetic field. Since this direct approach becomes rather awkward for general electromagnetic fields we choose a different more intuitive method which relies on quantum mechanical concepts [5]. We start from an effective Hamiltonian describing the interaction of particles with waves, and of waves with themselves. From this one derives the matrix element for the scattering process under consideration. From a detailed balance relation one easily obtains the Fokker-Planck equation. The result is compared with the equation for the mean distribution function derived in the usual weak turbulence expansion from the Vlasov equation, and two applications are briefly considered.

2. Derivation of the Diffusion Coefficient

The dynamics of a system of interacting particles and fields can be described by an effective Hamiltonian (we only write the interaction part of H):

$$\begin{aligned} H = & \sum_{\alpha} \int \mathrm{d}^3 p \, \mathrm{d}^3 k \left(\frac{1}{c} \boldsymbol{j}^{\alpha}(\boldsymbol{p}, \boldsymbol{k}) \cdot \boldsymbol{A}(k) - \varrho^{\alpha}(\boldsymbol{p}, \boldsymbol{k}) \Phi(\boldsymbol{k}) \right) \\ & + \int \mathrm{d}^3 k_1 \, \mathrm{d}^3 k_2 \, \mathrm{d}^3 k_3 \, V_{ijl} (\boldsymbol{k}_1, \dots, \boldsymbol{k}_3) \, E^i_{\boldsymbol{k}_4} E^j_{\boldsymbol{k}_2} E^l_{\boldsymbol{k}_3} \\ & + \int \mathrm{d}^3 k_1, \dots, \mathrm{d}^3 k_4 \, U_{ijlm} (\boldsymbol{k}_1, \dots, \boldsymbol{k}_4) \, E^i_{\boldsymbol{k}_1} E^j_{\boldsymbol{k}_2} E^l_{\boldsymbol{k}_3} E^m_{\boldsymbol{k}_4} \\ & + \quad \text{c. c.} \qquad\qquad + \dots \end{aligned} \tag{1}$$

where $\boldsymbol{j}^{\alpha}(\boldsymbol{p}, \boldsymbol{k})$, $\varrho^{\alpha}(\boldsymbol{p}, \boldsymbol{k})$ are the current and charge density operators of particles of type α, while V, U, etc. are the coupling coefficients of 3 waves,

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[1] B. B. KADOMTSEV, Plasma Turbulence, Academic Press, London and New York 1965.

[2] By "resonant" we always mean $\omega - \quad \cdot \quad = 0$, all other processes are called "nonresonant".

[3] P. A. STURROCK, Phys. Rev. **141**, 186 [1966].

[4] A. B. MACMAHON and W. E. DRUMMOND, Phys. Fluids **10**, 1714 [1967].

[5] After completion of this work we have learnt that a similar quantum mechanical treatment has been given by ROSS [6] and HARRIS [7].

4 waves, etc., appearing in the equation of motion for the waves:

$$\mathrm{DE} = \mathrm{VEE} + \mathrm{UEEE} + \ldots \tag{2}$$

(in compact notation). Eq. (1) is not written in a relativistically covariant form, which would not be very practical in cases where general (transverse and longitudinal) fields occur. We will only use the electric field ($\boldsymbol{B}$ being given by Faraday's law)

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \Phi .$$

In a medium with dielectric tensor ε_{ij} the modes E_k are eigensolutions of the system

$$D\,\boldsymbol{a} \equiv \left[\varepsilon_{ij} - \frac{c^2}{\omega^2}\,(k^2\,\delta_{ij} - k_i\,k_j)\right] a_j = 0 \tag{3}$$

where ω is a solution of $\det\{D\} = 0$, and $\boldsymbol{a}$ is the polarization vector, $\boldsymbol{E} = |E|\,\boldsymbol{a}$.

We are interested in the second order scattering of particles by waves. The matrix element is obtained from the Hamiltonian (1) and can be written in familiar diagram language:

$M =$  (4)

The first two diagrams describe particle scattering by absorption and re-emission of a free wave, the third diagram represents the scattering of a particle by a virtual wave, produced by the interaction of two other waves.

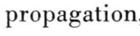
We will now, in a more intuitive way, give rules how to construct the algebraic expression of a diagram (of course this can also be done in a rigorous way). Since the number of wave quanta is defined by

$$N_k = |\boldsymbol{a}^* \cdot (\partial D/\partial\omega) \cdot \boldsymbol{a}|\,|E_k|^2/8\,\pi \tag{5}$$

an external wave line is represented by

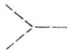
$$\frac{a_i}{\sqrt{2}}\,\frac{8\,\pi}{\sqrt{\boldsymbol{a}^* \cdot (\partial D/\partial\omega) \cdot \boldsymbol{a}}}\,. \tag{6}$$

An internal wave line is connected with free wave propagation, and will be represented by



$$D_{ij}^{-1}\,(\boldsymbol{k}, \omega)\,. \tag{7}$$

A three wave vertex is given by



$$2\,V_{ijl}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \tag{8}$$

(the factor 2 follows from symmetry considerations). In the classical limit the vertex contribution from the first term of (1) can be obtained from the expression [8]

$$e\left(\frac{\boldsymbol{v}}{c} \cdot \boldsymbol{A} - \Phi\right) = -i\,e\,E_i\,\gamma_{ij}\,\frac{k_j}{|k|} \tag{9}$$

where we defined

$$\gamma_{ij} = \gamma_{ji} \equiv \frac{1}{|k|\,\omega}\,(v_i\,k_j + (\omega - \boldsymbol{k}\cdot\boldsymbol{v})\,\delta_{ij})\,. \tag{10}$$

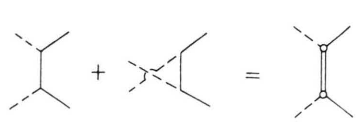
We can now identify

$$-i\,e\,\gamma_{ij}\,\frac{k_j}{|k|} = -i\,e\left(v_i + \frac{\omega - k\cdot v}{k^2}\,k_i\right)\frac{1}{\omega} \tag{11}$$

The factor $k_j/|k|$ in (11) represents the particle line being connected to the vertex (another factor $\exp\{-i\,\boldsymbol{k}\cdot(\boldsymbol{x} + \boldsymbol{v}\,t)\}$ connected to an overall particle line crossing the diagram, and other phase factors are omitted here). An internal particle line would correspond to a factor $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1}$. However there appears a certain difficulty, when giving a classical expression for a symmetrized diagram ("exchange pair"), as the first two in (4), since taken separately, both diagrams diverge in the limit $\hbar \to 0$, only their sum being a finite expression. One therefore has to take the limit $\hbar \to 0$ of the sum, which leads to a deriative,

$$\frac{\partial}{\partial \boldsymbol{v}}\,(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1}.$$

Formally one can replace the two diagrams by a symmetric one

 (12)

by intreoducing a new particle propagator:

$$\frac{|k|\,|k'|}{m\,(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)\,(\boldsymbol{k}'\cdot\boldsymbol{v} - \omega')}\,\delta_{ij}\,. \tag{13}$$

The matrix element M for the scattering of a wave $(\boldsymbol{k}, \boldsymbol{a})$ into a wave $(\boldsymbol{k}', \boldsymbol{a}')$ by a particle with

[6] D. W. Ross, Phys. Fluids **12**, 613 [1969].

[7] E. G. Harris, Advances in Plasma Physics, Vol. III, John Wiley & Sons Inc., New York 1970.

[8] We use Coulomb gauge $\boldsymbol{k}\cdot\boldsymbol{A} = 0$. The final expression is of course gauge invariant.

velocity $\boldsymbol{v}$ can now be written down immediately:

$$M_{kk'}^{aa'} = 4\pi \frac{1}{(\boldsymbol{a}^* \cdot (\partial D/\partial\omega)\cdot \boldsymbol{a})^{1/2}} \frac{1}{(\boldsymbol{a}^* \cdot (\partial D/\partial\omega')\cdot \boldsymbol{a})^{1/2}} \times \left[\frac{e^2}{m} \frac{a_i' \gamma_{ij}' \gamma_{jl} a_l}{(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)^2} - 2\, e\, i\, a_i' a_j V_{ijl} \frac{D_{lm}^{-1}(\boldsymbol{k}'', \omega'')}{\omega''^2} \gamma_{mn}'' \frac{k_n''}{|k''|} \right], \tag{14}$$

$$\omega'' = \omega - \omega', \qquad \boldsymbol{k}'' = \boldsymbol{k} - \boldsymbol{k}'.$$

The coupling coefficients defined by Eq. (2) can be obtained by solving the Vlasov equation (of the medium particles, not the test particles) iteratively as done in Ref. 1, p. 43. The result is [9]:

$$V_{ijl}(k, k', k'') = \frac{1}{2i} \sum_\alpha \frac{4\pi e_\alpha^3}{m_\alpha^2} \frac{|k'|\,|k''|}{\omega\,\omega'\,\omega''} \int \mathrm{d}u^3 \frac{v_i}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega} \times \left[\gamma_{jm}(\boldsymbol{k}') \frac{\partial}{\partial u_m} \frac{\gamma_{ln}(\boldsymbol{k}'')}{\boldsymbol{k}''\cdot\boldsymbol{v}-\omega''} \frac{\partial F}{\partial u_n} + \gamma_{ln}(\boldsymbol{k}'') \frac{\partial}{\partial u_n} \frac{\boldsymbol{k}'\cdot\boldsymbol{v}-\omega'}{\gamma_{jm}(\boldsymbol{k}')} \frac{\partial F}{\partial u_m} \right]. \tag{15}$$

In case the first order wave-particle and wave-wave scattering processes are not forbidden,

i. e. $\quad \omega - \boldsymbol{k}\cdot\boldsymbol{v} = 0, \quad$ or $\quad \omega - \omega' - \omega'' = 0$

for some $\boldsymbol{v}, \boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''$, singularities would arise from the factors $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-2}$, $D^{-1}(\boldsymbol{k}'', \omega'')$ in Eq. (14). They correspond to intermediate states lying on the energy shell. As is clear from quantum-mechanical scattering theory, these states have to be excluded from the summation over intermediate states which implies that $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-2}$, $D^{-1}(\boldsymbol{k}'', \omega'')$ have to be treated as principal values.

The transition probability for the process under consideration is given by

$$W_{\boldsymbol{kk}'}^{aa'} = \; = |M_{\boldsymbol{kk}'}^{aa'}|^2\, 2\pi\, \delta(\boldsymbol{k}''\cdot\boldsymbol{v} - \omega'')$$

(the δ-function comes from overall energy conservation). Following the arguments of TSYTOVICH [10, 11] we obtain the Fokker-Planck equation for the diffusion of particles α in a turbulent medium ($N_k \gg 1$):

$$\frac{\partial f_\alpha(u)}{\partial t} = \frac{\partial}{\partial m\,\boldsymbol{u}} \cdot \sum_{a,a'} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} N_{\boldsymbol{k}}^a N_{\boldsymbol{k}'}^{a'} W_{\boldsymbol{kk}'}^{aa'} \boldsymbol{k}'\boldsymbol{k}'' \cdot \frac{\partial f_\alpha(u)}{\partial m\,\boldsymbol{u}}, \tag{16}$$

$$\boldsymbol{u} = \frac{1}{\sqrt{1 - v^2/c^2}}\, \boldsymbol{v}.$$

The transition probability W can also be obtained classically by considering the interaction of dressed particles with waves. This approach has been used by ROSENBLUTH, COPPI and SUDAN [11] for the case of the loss cone instability. The quantum mechanical approach, outlined above, might perhaps give some more understanding of the basic physical processes and is more suited for calculating higher order terms. The relative importance of the two terms in Eq. (14) depends on the special problems under consideration. Two different cases will be discussed in Section 4.

3. Comparison with Conventional Weak Turbulence Theory

The weak turbulence expansion starting from the Vlasov equation, as summarized in Ref. [1], also gives an equation for the change of the mean distribution function (Ref. [1], Eq. II, 61). However, one immediately realizes that this is not a diffusion equation but contains derivatives of $\boldsymbol{v}$ up to fourth order:

It can be written in the form:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \Delta v\, f + \frac{1}{2} \frac{\partial^2}{\partial v^2} \Delta v^{(2)} f + \frac{1}{3!} \frac{\partial^3}{\partial v^3} \Delta v^{(3)} f + \frac{1}{4!} \frac{\partial^4}{\partial v^4} \Delta v^{(4)} f. \tag{17}$$

The coefficients $\Delta v^{(3)}, \Delta v^{(4)}$ are proportional to $\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$, $\delta(\omega - \omega' - \omega'')$ originating from the first order resonances. In general the higher order derivatives in Eq. (17) no longer garanty the positivity of f and have hence to be considered with some suspicion. It seems to be a conceptual advantage of the quantum mechanical method used above, that no terms $\sim \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$ etc. appear, since the corresponding denominators $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1}$ etc. in the matrix element have to be treated as principal values, M being proportional to $\delta(\omega'' - \boldsymbol{k}''\cdot\boldsymbol{v})$ because of energy conservation.

But also if the resonant contributions $\sim \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$ etc. are neglegible, such that the $\Delta v^{(3)}, \Delta v^{(4)}$ vanish in (17), this equation is not acceptable physically since the diffusion coefficient $\frac{1}{2}\Delta v^{(2)}$ is not positive definite. While according to Eq. (14) the diffusion

[9] Despite its unsymmetric appearance this expression is symmetric in all three variables $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}''$ and the respective tensor indices i, j, l.

[10] L. M. KOVRIZHNYKH and V. N. TSYTOVICH, JETP 19, 1494 [1964].

[11] M. N. ROSENBLUTH, B. COPPI, and R. N. SUDAN, in: Plasma Physics and Controlled Nuclear Fusion Research (International Atomic Energy Agency, Vienna 1969), Vol. I, p. 771.

coefficient consists of the absolute square of a sum of two terms:

$$|M_1+M_2|^2=|M_1|^2+2\,\mathrm{Re}(M_1 M_2^*)+|M_2|^2,$$

the last term is missing in Eq. II, 61, Ref. [1]. It can easily be seen, why the $|M_2|^2$-term is not obtained by the usual weak turbulence approach. From the Vlasov equation one derives the following relation for the mean distribution function:

$$\frac{\partial f}{\partial t}=\langle E^2\rangle A f+\langle E^3\rangle B f+\langle E^4\rangle C f+\dots \quad (18)$$

The usual statistical closure (quasi Gaussean approximation) only consists of replacing $\langle E^4\rangle$ by $\langle E^2\rangle^2$ (corresponding to $|M_1|^2$), and inserting the equation of motion for E, Eq. (2), into one of the factors of $\langle E^3\rangle$, thus providing another term $\langle E^2\rangle^2$ (corresponding to $2\,\mathrm{Re}\,M_1 M_2^*$). In principle the missing term $|M_2|^2$ could be obtained by inserting Eq. (2) simultaneously into both factors of $\langle E^2\rangle$ in the first term of (18). Since however terms which appear when substituting in $\langle E^2\rangle$ one factor E by UE^3, should not be taken into account, this substitution program looks somewhat artificial.

Using a simple quantum mechanical picture, we have derived a Fokker-Planck equation for the non-resonant diffusion of charged particles in (weakly) turbulent electric and magnetic fields. The result does not have the deficiencies of the corresponding equation obtained by the conventional weak turbulence approach (see Ref. [1], Eq. II, 61): it is a proper diffusion equation, not containing higher order derivatives and the diffusion coefficient is automatically positive definite.

4. Applications

A number of different processes are described by the matrix element (14). Here we want to treat briefly two cases:

a) Acceleration of ions by turbulent ion sound waves

The scattering of ions by ion sound waves plays a fundamental role in the dynamics of the ion sound instability, since nonlinear Landau damping by ions is believed to be the essential stabilization mechanism. It is however well known that assuming a Maxwellian distribution function for the ions the nonlinear damping rate is too small, giving rise to a turbulent energy $W=E^2/8\pi n T$ too high as compared with the experimentally observed values. It was therefore recently suggested (Sagdeev, private communication) that the ion distribution forms a high energy tail, which would enhance nonlinear Landau damping. This process should be described by the Fokker-Planck Eq. (16), which in this case has the form:

$$\frac{\partial f_i}{\partial t}=\frac{\partial}{\partial m_i \boldsymbol{v}}\cdot\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} \times\frac{1}{4}\left|\frac{e^2}{m_i}\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2}+\frac{e}{k''^2\,\varepsilon(\boldsymbol{k}'')}\,2\,V_{kk'}\right|^2 \quad (19)$$
$$\times|\Phi_{\boldsymbol{k}}|^2\,|\Phi_{\boldsymbol{k}}|^2\,2\pi\,\delta(\boldsymbol{k}''\cdot\boldsymbol{v}-\omega'')\,\boldsymbol{k}'\boldsymbol{k}''\cdot\frac{\partial f_i}{\partial m_i\boldsymbol{v}}\,.$$

Since $T_e/T_i \gg 1$, $\omega/k \gg v_i$ for ion sound turbulence to be possible, we find that the two terms of $V_{\boldsymbol{kk}'}$, $V_{\boldsymbol{kk}'}=V_{\boldsymbol{kk}'}^{(i)}+V_{\boldsymbol{kk}'}^{(e)}$, are approximately:

$$\frac{e}{k''^2\,\varepsilon(\boldsymbol{k}'',\omega'')}\,2\,V^{(i)}_{\boldsymbol{kk}'}\approx-\frac{e^2}{m_i}\frac{1}{k''^2} \quad (20)$$
$$\times\left(\frac{k'^2\,\boldsymbol{k}\cdot\boldsymbol{k}''}{\omega'^2}-\frac{k^2\,\boldsymbol{k}'\cdot\boldsymbol{k}''}{\omega^2}+\frac{2\,\boldsymbol{k}''\cdot\boldsymbol{k}''\,\boldsymbol{k}'\cdot\boldsymbol{k}''}{\omega\,\omega'}\right)$$

and

$$\frac{e}{k''^2\,\varepsilon(\boldsymbol{k}''\,\omega'')}\,2\,V^{(e)}_{\boldsymbol{kk}'}\approx\frac{e^2}{m_i}\frac{1}{k''^2\lambda_D^2}\frac{1}{c_s^2}\frac{\omega''^2}{\omega_{pi}^2}\,. \quad (21)$$

For thermal particles, $v\sim v_i$, one has

$$\omega''^2/\omega^2\sim T_i/T_e\,.$$

The electron contribution is small and the ion term $\approx -e^2/m_i\cdot\boldsymbol{k}\cdot\boldsymbol{k}'/\omega^2$ just cancels the first term in the matrix element in Eq. (19) (This is similar to the case of electron diffusion by Langmuir waves discussed by MACMAHON and DRUMMOND [4]). Thus for the bulk of the particles the diffusion is rather slow, the heating rate being of the order:

$$dT_i/dt\sim\omega_{pi}W^2 T_i\,,\quad W=\tilde{E}^2/8\pi n T\,. \quad (22)$$

Comparing Eq. (22) with the second order adiabatic diffusion due to the nonstationarity of the spectrum

$$\frac{\partial f_i}{\partial t}\approx\frac{e^2}{m_i^2}\frac{\partial}{\partial v}\int\frac{d^3k}{(2\pi)^3}\frac{|\Phi_{\boldsymbol{k}}|^2\,\gamma\,\boldsymbol{k}\,\boldsymbol{k}}{\omega^2}\frac{\partial f}{\partial v}\,.$$
$$\frac{\partial|\Phi_{\boldsymbol{k}}|^2}{\partial t}=2\gamma\,|\Phi_{\boldsymbol{k}}|^2,$$

we see that the fourth order process becomes important for $\gamma<\omega_{pi}W\,T_i/T_e$.

Superthermal particles however, $v\gg v_i$, are diffusing much faster. For values of v still small compared to ω/k Eq. (19) is approximately:

$$\frac{\partial f_i}{\partial t} \approx 2\pi \frac{e^4}{m_i^4} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{(\boldsymbol{k}\cdot\boldsymbol{k}')^2}{\omega^4} \frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{\omega^2} |\Phi_{\boldsymbol{k}}|^2 |\Phi_{\boldsymbol{k}'}|^2 \times \boldsymbol{k}' \boldsymbol{k}'' \delta(\omega'' - \boldsymbol{k}''\cdot\boldsymbol{v}) \frac{\partial f}{\partial \boldsymbol{v}} \tag{23}$$

$$\approx \omega_{\mathrm{pi}} W \frac{\partial}{\partial v} v^2 \frac{\partial f}{\partial v}.$$

It is apparent that this leads to the formation of a high energy tail.

b) Diffusion of electrons in a radiation field

As a second application of Eq. (16) we want to discuss briefly the scattering of electrons by transverse fields. We assume an isotropic medium and sum over the two transversal polarization states using the relation:

$$\sum_{\nu=1,2} a_i^{(\nu)} a_j^{(\nu)} = \delta_{ij} - \frac{1}{k^2} k_j k_i . \tag{24}$$

For relativistic particle energies and densities such that $\omega_{\mathrm{pe}} \sim k c$, both terms in M, Eq. (14) are important. Explicit formulas can be obtained in a straightforward way. Here we only want to give the result in the nonrelativistic limit where the collective term M_2 is negligible. The first term reduces to:

$$D_{\mathrm{NR}} = \frac{e^4}{m_e^4} 4\pi^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1 + \frac{(\boldsymbol{k}\,\boldsymbol{k}')^2}{k'^2 k^2}}{\omega^4} \boldsymbol{k}' \boldsymbol{k}'' \frac{|E_k|^2}{4\pi} \frac{|E_{k'}|^2}{4\pi} \times 2\pi\,\delta(\omega-\omega') \tag{25}$$

which for isotropic radiation becomes:

$$D_{\mathrm{NR}} = \frac{m_e^4}{e^4} \frac{1}{3} (4\pi)^2 \times \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{k^2 + k'^2}{\omega^4} \frac{|E_k|^2}{4\pi} \frac{|E_{k'}|^2}{4\pi} \times 2\pi\,\delta(\omega-\omega'). \tag{26}$$

This is the expression given by DREICER [12] (see also Ref. [13]).

Conclusions:

Thus the quantum mechanical concept leading to Eq. (16) seems to be advantageous in a twofold way. First it eliminates in a natural way the resonant contributions $\sim \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$, $\delta(\omega-\omega'-\omega'')$ thus yielding a proper diffusion equation and second it provides for a positive definite diffusion coefficient, eliminating an (as yet unnoticed) error of the usual weak turbulence approach. We have not incorporated a static magnetic field (thus treating the case $\beta = 8\pi n T/B^2 \gg 1$, while MACMAHON and DRUMMOND [4] have considered the opposite case $\beta \ll 1$). In the presence of a magnetic field the spatial diffusion perpendicular to $\boldsymbol{B}_0$ is an interesting quantity. In the special case that the drift approximation for the particle orbits is applicable ($\omega < \Omega_c$, $k\varrho_c < 1$) the simple relation between parallel diffusion in velocity space and transverse spatial diffusion given by DRUMMOND and ROSENBLUTH [14] is correct for arbitrary β: replace

$$\frac{e}{m} E_{\parallel}(k) \text{ by } c\frac{E_\perp}{B_0},$$

since the ∇B_1-drift v_B is small compared to the $E_1 \times B_0$-drift v_E, $v_B/v_E = k\varrho_c \cdot v_{th}/\omega/k \ll 1$. When the drift approximation is not applicable the whole analysis has to be carried out using helical unperturbed orbits.

[12] H. DREICER, Phys. Fluids 5, 735 [1964].
[13] D. BISKAMP and D. PFIRSCH, Z. Naturforsch. **22a**, 145 [1967].
[14] W. E. DRUMMOND and M. N. ROSENBLUTH, Phys. Fluids 5, 1507 [1962].